

# Advanced Mathematical Methods with Maple

Derek Richards



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# 8

## Fourier series and systems of orthogonal functions

### 8.1 Introduction

In chapter 4 we introduced Taylor's series, a representation in which the function is expressed in terms of the value of the function and all its derivatives at the point of expansion. In general such series are useful only close to the point of expansion, and far from here it is usually the case that very many terms of the series are needed to give a good approximation. An example of this behaviour is shown in figure 8.1, where we compare various Taylor polynomials of  $\sin x$  about  $x = 0$  over the range  $0 < x < 2\pi$ .

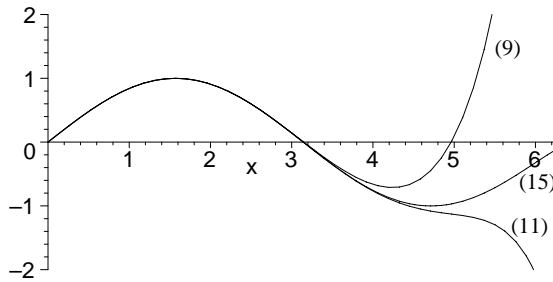


Figure 8.1 Taylor polynomials of  $\sin x$  with degree 9, 11 and 15.

It is seen that many terms are needed to obtain a good approximation at  $x = 2\pi$ . Such behaviour should not be surprising, for the  $n$ th term of the Taylor's series is  $(-1)^n x^{2n+1}/(2n+1)!$  and at  $x = 2\pi$  this is larger than 1 if  $n \leq 6$ . Some values of the magnitude of this coefficient for various  $n$  at  $x = 2\pi$  and  $3\pi$  are shown in figure 8.2.

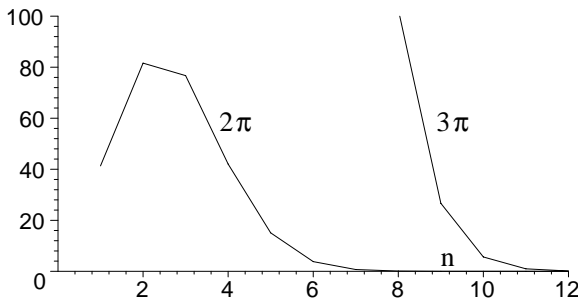


Figure 8.2 Dependence of  $x^{2n+1}/(2n+1)!$ , at  $x = 2\pi$  and  $3\pi$ , upon  $n$ .

A Taylor's series approximation is most accurate near the point of expansion,  $x = a$ , and its accuracy generally decreases as  $|x - a|$  increases, so this type of approximation suffers from the defect that it is not usually uniformly accurate over the required range of  $x$ , although the Padé approximant of the Taylor's series, introduced in chapter 6, can often provide improved accuracy.

Fourier series eliminate these problems by approximating functions in a quite different manner. The essential idea is very simple. Suppose we have a set of functions  $\phi_k(x)$ ,  $k = 1, 2, \dots$ , (which may be complex) defined on some interval  $a \leq x \leq b$ ; these could be, for instance, the polynomials  $\phi_k(x) = x^k$ , or the trigonometric functions  $\phi_k(x) = \sin kx$ . Then we attempt to approximate a function  $f(x)$  as the sum

$$f(x) \simeq f_N(x) = \sum_{k=1}^N c_k \phi_k(x)$$

by choosing the coefficients  $c_k$  to minimise the mean square difference

$$E_N = \int_a^b dx \left| f(x) - \sum_{k=1}^N c_k \phi_k(x) \right|^2.$$

This is a more democratic method of approximation because no point in the interval is picked out for favoured treatment, as in a Taylor's series. In order to put this idea into practice we need to know how to choose the functions  $\phi_k(x)$  and to understand how the approximation converges with increasing  $N$ . For this it is necessary to introduce the notion of a complete set of functions together with some connected technical details.

### Exercise 8.1

Use Stirling's approximation to show that the modulus of the  $n$ th term of the Taylor's series of  $\sin x$ ,  $x^{2n+1}/(2n+1)!$ , is small for  $n > ex/2$ , and that for large  $|x|$  the largest term in the Taylor's series has a magnitude of about  $e^x/\sqrt{2\pi x}$ .

In addition show that with arithmetic accurate to  $N$  significant figures the series approximation for  $\sin x$  can be used directly to find values of  $\sin x$  accurate to  $M(< N)$  significant figures only for  $x < (N - M) \ln 10$ . Check this behaviour using Maple.

### Exercise 8.2

This exercise is about the approximation of  $\sin x$  on the interval  $(0, \pi)$  using the functions  $\phi_1 = 1$ ,  $\phi_2 = x$  and  $\phi_3 = x^2$ .

Use Maple to evaluate the integral

$$E(a, b, c) = \int_0^\pi dx (a + bx + cx^2 - \sin x)^2$$

to form the function  $E(a, b, c)$  which is quadratic in  $a$ ,  $b$  and  $c$ . Find the position of the minimum of this function by solving the three equations  $\frac{\partial E}{\partial a} = 0$ ,  $\frac{\partial E}{\partial b} = 0$  and  $\frac{\partial E}{\partial c} = 0$  for  $a$ ,  $b$  and  $c$ . Hence show that

$$\sin x \simeq 12 \frac{\pi^2 - 10}{\pi^3} - 60 \frac{\pi^2 - 12}{\pi^5} x(\pi - x), \quad 0 \leq x \leq \pi.$$

Compare your approximation graphically with the exact function and various Taylor polynomials.

Note that with Maple it is relatively easy to include higher order polynomials in this expansion: it is worth exploring the effect of doing this.

## 8.2 Orthogonal systems of functions

Here we consider complex functions of the real variable  $x$  on an interval  $a \leq x \leq b$ . The *inner product* of two such functions  $f(x)$  and  $g(x)$  is denoted by  $(f, g)$  and is defined by the integral

$$(f, g) = \int_a^b dx f^*(x)g(x), \quad (8.1)$$

where  $f^*$  denotes the complex conjugate of  $f$ ; note that  $(g, f) = (f, g)^*$ . The inner product of a function with itself is real, positive and  $\sqrt{(f, f)}$  is named the *norm*. A function whose norm is unity is said to be *normalised*.

Two functions  $f$  and  $g$  are *orthogonal* if their inner product is zero,  $(f, g) = 0$ . A system of normalised functions  $\phi_k(x)$ ,  $k = 1, 2, \dots$ , every pair of which is orthogonal is named an *orthogonal system*. If, in addition, each function is normalised, so

$$(\phi_r, \phi_s) = \delta_{rs} = \begin{cases} 1, & r = s, \\ 0, & r \neq s, \end{cases}$$

the system is called an *orthonormal system*. The symbol  $\delta_{rs}$  introduced here is named the *Kronecker delta*. An example of a real orthonormal system on the interval  $(0, 2\pi)$ , or more generally any interval of length  $2\pi$ , is the set of functions

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{\cos x}{\sqrt{\pi}}, \quad \frac{\cos 2x}{\sqrt{\pi}}, \quad \dots, \quad \frac{\cos kx}{\sqrt{\pi}}, \quad \dots$$

On the same interval the set of complex functions

$$\phi_k(x) = \frac{e^{ikx}}{\sqrt{2\pi}}, \quad k = 0, \pm 1, \pm 2, \dots$$

is also an orthonormal system.

### Exercise 8.3

Find the appropriate value of the constant  $A$  that makes the norm of each of the functions

$$\phi_1(x) = Ax, \quad \phi_2(x) = A(3x^2 - 1), \quad \phi_3(x) = Ax(5x^2 - 3)$$

on the intervals  $-1 \leq x \leq 1$  and  $0 \leq x \leq 1$ , unity. For each interval determine the matrix of inner products  $(\phi_i, \phi_j)$  for  $i, j = 1, 2, 3$ .

### Exercise 8.4

By evaluating  $(h, h)$ , where  $h(x) = (f, g)f(x) - (f, f)g(x)$ , and using the fact that  $(h, h) \geq 0$ , prove the Schwarz inequality,

$$|(f, g)|^2 \leq (f, f)(g, g), \quad (8.2)$$

**Exercise 8.5**

Show that the functions

$$\phi_k(x) = \sqrt{\frac{2}{\alpha}} \sin\left(\frac{2\pi}{\alpha} kx\right), \quad k = 1, 2, \dots,$$

are orthogonal on any interval of length  $\alpha$ .

**8.3 Expansions in terms of orthonormal functions**

Suppose that  $\phi_k(x)$ ,  $k = 1, 2, \dots$ , is a system of orthonormal functions on the interval  $a \leq x \leq b$ , then we expand a given real function  $f(x)$  in terms of these functions by writing

$$f(x) \simeq f_N(x) = \sum_{k=1}^N c_k \phi_k(x), \quad a \leq x \leq b, \quad (8.3)$$

and choose the  $N$  coefficients  $c_k$  — which may be complex — to minimise the square of the norm of  $(f_N - f)$ . That is, we minimise the function

$$\begin{aligned} F(\mathbf{c}) &= \int_a^b dx \left| \sum_{k=1}^N c_k \phi_k(x) - f(x) \right|^2 \\ &= \sum_{k=1}^N |c_k - (\phi_k, f)|^2 - \sum_{k=1}^N |(\phi_k, f)|^2 + (f, f). \end{aligned} \quad (8.4)$$

It follows from this last expression that  $F(\mathbf{c})$  has its only minimum when the first term is made zero by choosing

$$c_j = (\phi_j, f), \quad j = 1, 2, \dots, N. \quad (8.5)$$

The numbers  $c_j = (\phi_j, f)$  are the expansion coefficients of  $f$  with respect to the orthogonal system  $\{\phi_1, \phi_2, \dots\}$ . This type of approximation is called an *approximation in the mean*.

An important consequence follows from the trivial observation that

$$\int_a^b dx \left| \sum_{k=1}^N c_k \phi_k(x) - f(x) \right|^2 \geq 0,$$

and by expanding the integrand and integrating term by term to give

$$\sum_{k=1}^N |c_k|^2 - \sum_{k=1}^N [c_k (\phi_k, f)^* + c_k^* (\phi_k, f)] + (f, f) \geq 0, \quad \text{for any } c_k.$$

Hence

$$\sum_{k=1}^N |c_k|^2 \leq (f, f) \quad \text{if} \quad c_k = (\phi_k, f) \quad [\text{and} \quad (\phi_k, \phi_k) = 1].$$

Since  $(f, f)$  is independent of  $N$  it follows that

$$\sum_{k=1}^{\infty} |c_k|^2 \leq (f, f), \quad \text{Bessel's inequality.} \quad (8.6)$$

Bessel's inequality is true for every orthogonal system: it shows that the sum of squares of the coefficients always converges, provided the norm of  $f$  exists. From this inequality it follows from section 4.3.1 that  $c_k \rightarrow 0$  as  $k \rightarrow \infty$ .

### Exercise 8.6

Derive equation 8.4.

### Exercise 8.7

Equations 8.5, for the expansion coefficients, and Bessel's inequality, equation 8.6, were both derived for an orthonormal system for which  $(\phi_j, \phi_k) = \delta_{jk}$ . Show that if the  $\phi_k$  are orthogonal, but not necessarily orthonormal, these relations become

$$c_j = \frac{(\phi_j, f)}{(\phi_j, \phi_j)} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{|(\phi_k, f)|^2}{(\phi_k, \phi_k)} \leq (f, f).$$

## 8.4 Complete systems

A *complete orthogonal system*,  $\phi_k(x)$ ,  $k = 1, 2, \dots$ , has the property that any function, taken from a given particular set of functions, can be approximated in the mean to any desired accuracy by choosing  $N$  large enough. In other words, for any  $\epsilon > 0$ , no matter how small, we can find an  $N(\epsilon)$  such that for  $M > N(\epsilon)$

$$\int_a^b dx \left| f(x) - \sum_{k=1}^M c_k \phi_k(x) \right|^2 < \epsilon, \quad \text{where} \quad c_k = \frac{(\phi_k, f)}{(\phi_k, \phi_k)}. \quad (8.7)$$

That is, the mean square error can be made arbitrarily small. Notice that this definition needs the function  $f$  to belong to a given set, normally the set of integrable functions. For a complete orthogonal system it can be proved that Bessel's inequality becomes the equality

$$(f, f) = \sum_{k=1}^{\infty} (\phi_k, \phi_k) |c_k|^2 = \sum_{k=1}^{\infty} \frac{(\phi_k, f)^2}{(\phi_k, \phi_k)}, \quad (8.8)$$

and this is known as the *completeness relation*. It may also be shown that a sufficient condition for an orthogonal system to be complete is that this completeness relation holds; a proof of this statement is given in Courant and Hilbert (1965, page 52).

There are three points worthy of note.

- First, the functions  $f(x)$  being approximated need not be continuous or differentiable at every point in the interval of approximation, as required by Taylor's series.

- Second, the fact that

$$\lim_{N \rightarrow \infty} \int_a^b dx \left| f(x) - \sum_{k=1}^N c_k \phi_k(x) \right|^2 = 0, \quad c_k = \frac{(\phi_k, f)}{(\phi_k, \phi_k)}, \quad (8.9)$$

does *not* imply pointwise convergence, that is,

$$\lim_{N \rightarrow \infty} \sum_{k=1}^N c_k \phi_k(x) = f(x), \quad a \leq x \leq b. \quad (8.10)$$

If the limit 8.9 holds we say that the sequence of functions

$$f_N(x) = \sum_{k=1}^N c_k \phi_k(x)$$

converge to  $f(x)$  in the mean. If the limit 8.10 holds  $f_N(x)$  converges pointwise to  $f(x)$ , section 4.6. If, however, the series  $f_N(x)$  converges uniformly then convergence in the mean implies pointwise convergence.

- Third, if two piecewise continuous functions have the same expansion coefficients with respect to a complete system of functions then it may be shown that they are identical, see for example Courant and Hilbert (1965, page 54).

Finally, we note that systems of functions can be complete even if they are not orthogonal. Examples of such complete systems are the polynomials  $\phi_k(x) = x^k$ ,

$$1, x, x^2, \dots, x^n, \dots,$$

which form a complete system in any closed interval  $a \leq x \leq b$ , for the approximation theorem of Weierstrass states that any function continuous in the interval  $a \leq x \leq b$  may be approximated uniformly by polynomials in this interval. This theorem asserts uniform convergence, not just convergence in the mean, but restricts the class of functions to be continuous. A proof may be found in Powell (1981, chapter 6).

Another set of functions is

$$\frac{1}{x + \lambda_1}, \quad \frac{1}{x + \lambda_2}, \quad \dots, \frac{1}{x + \lambda_n}, \dots,$$

where  $\lambda_1, \lambda_2, \dots, \lambda_n, \dots$  are positive numbers which tend to infinity with increasing  $n$ ; this set is complete in every finite positive interval. An example of the use of these functions is given in exercise 8.28 (page 294), and another set of complete functions is given in exercise 8.41 (page 298).

In this and the preceding sections we have introduced the ideas of:

- inner product and norm;
- orthogonal functions;
- complete orthonormal systems.

You should ensure that you understand these ideas before passing to the next section.

### 8.5 Fourier series

In modern mathematics a Fourier series is an expansion of a function in terms of a set of complete functions. Originally and in many modern texts the same name is used in the more restrictive sense to mean an expansion in terms of the trigonometric functions

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \dots, \cos nx, \sin nx, \dots \quad (8.11)$$

or their complex equivalents

$$\phi_k = e^{-ikx}, \quad k = 0, \pm 1, \pm 2, \dots, \quad (8.12)$$

which are complete and orthogonal on the interval  $(-\pi, \pi)$ , or any interval of length  $2\pi$ ; the interval  $(0, 2\pi)$  is often used. Series of this type are named *trigonometric series* if it is necessary to distinguish them from more general Fourier series: in the remainder of this chapter we treat the two names as synonyms. Trigonometric series are one of the simplest of this class of Fourier expansions — because they involve the well understood trigonometric functions — and have very many applications.

Any sufficiently well behaved function  $f(x)$  may be approximated by the *trigonometric series*

$$F(x) = \sum_{k=-\infty}^{\infty} c_k e^{-ikx}, \quad c_k = \frac{(\phi_k, f)}{(\phi_k, \phi_k)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx e^{ikx} f(x), \quad (8.13)$$

where we have used the first result of exercise 8.7. We restrict our attention to real functions, in which case  $c_0$  is real, and is just the mean value of the function  $f(x)$ , and  $c_{-k} = c_k^*$ . The constants  $c_k$  are named the *Fourier coefficients*.

The Fourier series  $F(x)$  is often written in the real form

$$F(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx, \quad (8.14)$$

where  $a_k = 2\Re(c_k)$ ,  $b_k = 2\Im(c_k)$ , or

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} dx f(x), \quad \begin{pmatrix} a_k \\ b_k \end{pmatrix} = \frac{1}{\pi} \int_{-\pi}^{\pi} dx \begin{pmatrix} \cos kx \\ \sin kx \end{pmatrix} f(x), \quad k = 1, 2, \dots \quad (8.15)$$

The constants  $a_k$  and  $b_k$  are also called Fourier coefficients. It is often more efficient and elegant to use the complex form of the Fourier series, though in special cases, see for instance exercise 8.10, the real form is more convenient.

One of the main questions to be settled is how the Fourier series  $F(x)$  relates to the original function  $f(x)$ . This relation is given by Fourier's theorem, discussed next, but before you read this it will be helpful to do the following two exercises.

#### Exercise 8.8

Show that the Fourier series of the function  $f(x) = |x|$  on the interval  $-\pi \leq x \leq \pi$  is

$$F(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2}.$$

Use Maple to compare graphically the  $N$ th partial sum,  $F_N(x)$ , of the above series with  $f(x)$  for  $N = 1, 2, 6$  and  $10$  over the range  $-2\pi \leq x \leq 2\pi$ .

Further, show that the mean square error, defined in equation 8.7, of the  $N$ th partial sum decreases as  $N^{-3}$ . Show also that

$$F_N(0) = \frac{1}{\pi N} + O(N^{-2}).$$

In this example we notice that even the first two terms of the Fourier series provide a reasonable approximation to  $|x|$  for  $-\pi \leq x \leq \pi$ , and for larger values of  $N$  the graphs of  $F_N(x)$  and  $|x|$  are indistinguishable over most of the range. A close inspection of the graphs near  $x = 0$ , where  $f$  has no derivative, shows that here more terms are needed to obtain the same degree of accuracy as elsewhere.

For  $|x| > \pi$ ,  $f(x)$  and  $F(x)$  are different. This is not surprising as  $F(x)$  is a periodic function with period  $\pi$  — generally this type of Fourier series is  $2\pi$ -periodic but here  $F(x)$  is even about  $x = \pi$ .

Finally, note that for large  $k$  the Fourier coefficients  $c_k$  are  $O(k^{-2})$ ; we shall see that this behaviour is partly due to  $f(x)$  being even about  $x = 0$ .

Now consider a function which is piecewise smooth on  $(-\pi, \pi)$  and discontinuous at  $x = 0$ .

### Exercise 8.9

Show that the Fourier series of the function

$$f(x) = \begin{cases} x/\pi, & -\pi < x < 0, \\ 1 - x/\pi, & 0 \leq x < \pi, \end{cases}$$

is

$$F(x) = \frac{4}{\pi^2} \sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1}.$$

Use Maple to compare graphically the  $N$ th partial sum of this series with  $f(x)$  for  $N = 1, 4, 10$  and  $20$ . Make your comparisons over the interval  $(-2\pi, 2\pi)$  and investigate the behaviour of the partial sums of the Fourier series in the range  $-0.1 < x < 0.1$  in more detail.

Find the values of  $f(0)$  and  $f(\pm\pi)$  and show that

$$F(0) = \frac{1}{2} \quad \text{and} \quad F(\pm\pi) = -\frac{1}{2}.$$

Hint: remember that the piecewise function can be defined in Maple using the command `x->piecewise(x<0, x/Pi, 1-x/Pi);`.

In this comparison there are four points to notice:

- First, observe that for  $-\pi < x < \pi$ , but  $x$  not too near 0 or  $\pm\pi$ ,  $F_N(x)$  is close to  $f(x)$  but that  $F(x) \neq f(x)$  at  $x = 0$  and  $\pm\pi$ .
- Second, as in the previous example,  $F$  and  $f$  are different for  $|x| > \pi$  because  $F(x)$  is a periodic extension of  $f(x)$ .
- Third, in this case the convergence of the partial sums to  $f(x)$  is slower because now  $c_k = O(k^{-1})$ ; again this behaviour is due to the nature of  $f(x)$ , as will be seen later.
- Finally, we see that for  $x \simeq 0$ ,  $F_N(x)$  oscillates about  $f(x)$  with a period that decreases with increasing  $N$  but with an amplitude that does not decrease: this important phenomenon is due to the discontinuity in  $f(x)$  at  $x = 0$  and will be discussed in section 8.11.

Some of the observations made above are summarised in the following theorem, which gives *sufficient* conditions for the Fourier series of a function to coincide with the function.

### Fourier's theorem

Let  $f(x)$  be a function given on the interval  $-\pi \leq x < \pi$  and defined for all other values of  $x$  by the equation

$$f(x + 2\pi) = f(x)$$

so that  $f(x)$  is  $2\pi$ -periodic. Assume that  $\int_{-\pi}^{\pi} dx f(x)$  exists and that the complex Fourier coefficients  $c_k$  are defined by the equations

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx f(x) e^{ikx}, \quad k = 0, \pm 1, \pm 2, \dots,$$

then if  $-\pi < a \leq x \leq b < \pi$  and if in this interval  $|f(x)|$  is bounded, the series

$$F(x) = \sum_{k=-\infty}^{\infty} c_k e^{-ikx} \quad (8.16)$$

is convergent and has the value

$$F(x) = \frac{1}{2} \left( \lim_{\epsilon \rightarrow 0_+} f(x + \epsilon) + \lim_{\epsilon \rightarrow 0_+} f(x - \epsilon) \right). \quad (8.17)$$

If  $f(x)$  is continuous at a point  $x = w$  the limit reduces to  $F(w) = f(w)$ .

The conditions assumed here are normally met by functions found in practical applications. In the example treated in exercise 8.9 we have, at  $x = 0$ ,

$$\lim_{\epsilon \rightarrow 0_+} f(\epsilon) = 1 \quad \text{and} \quad \lim_{\epsilon \rightarrow 0_+} f(-\epsilon) = 0,$$

so equation 8.17 shows that the Fourier series converges to  $\frac{1}{2}$  at  $x = 0$ , as found in the exercise.

Fourier's theorem gives the general relation between the Fourier series and  $f(x)$ . In addition it can be shown that if the Fourier coefficients have bounded variation and  $|c_k| \rightarrow 0$  as  $k \rightarrow \infty$  the Fourier series converges uniformly in the interval  $0 < |x| < \pi$ . At  $x = 0$  care is sometimes needed as the real and imaginary parts of the series behave differently; convergence of the real part depends upon the convergence of the sum  $c_0 + c_1 + c_2 + \dots$ , whereas the imaginary part is zero, since  $c_0$  is real for real functions (Zygmund, 1990, chapter 1).

The completeness relation, equation 8.8 (page 271), modified slightly because the functions used here are not normalised, gives the identity

$$\sum_{k=-\infty}^{\infty} |c_k|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx f(x)^2, \quad (8.18)$$

or, for the real form of the Fourier series,

$$\frac{1}{2} a_0^2 + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} dx f(x)^2. \quad (8.19)$$

These relations are known as *Parseval's theorem*. It follows from this relation that if the integral exists  $|c_k|$  tends to zero faster than  $|k|^{-1/2}$  as  $|k| \rightarrow \infty$ .

There is also a converse of Parseval's theorem: the Riesz–Fischer theorem, which states that if numbers  $c_k$  exist such that the sum in equation 8.18 exists then the series defined in equation 8.13 (page 273), is the Fourier series of a square integrable function. A proof of this theorem can be found in Zygmund (1990, chapter 4).

In the appendix to this chapter some common Fourier series are listed.

### Exercise 8.10

Show that if  $f(x)$  is a real function even about  $x = 0$  then its Fourier series, on the interval  $(-\pi, \pi)$ , contains only cosine terms, but that if  $f(x)$  is an odd function its Fourier series contains only sine terms.

### Exercise 8.11

Let  $F(x)$  be the Fourier series of the function  $f(x) = x^2$  on the interval  $0 \leq x \leq 2\pi$ . Sketch the graph of  $F(x)$  in the interval  $(-2\pi, 4\pi)$ . What are the values of  $F(2n\pi)$  for integer  $n$ ? Note, you are not expected to find an explicit form for  $F(x)$ .

### Exercise 8.12

Use the relation

$$\begin{aligned} \ln \left( \frac{1}{1-z} \right) &= z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \cdots + \frac{1}{n}z^n + \cdots, \quad |z| \leq 1, z \neq 1, \\ &= \ln \left| \frac{1}{1-z} \right| - i \arg(1-z) \end{aligned}$$

to show that for  $-\pi \leq \theta < \pi$ ,

$$\begin{aligned} \frac{1}{2} \ln \left( \frac{1}{1+r^2-2r \cos \theta} \right) &= \sum_{k=1}^{\infty} \frac{r^k}{k} \cos k\theta, \quad r \leq 1, \\ \tan^{-1} \left( \frac{r \sin \theta}{1-r \cos \theta} \right) &= \sum_{k=1}^{\infty} \frac{r^k}{k} \sin k\theta, \quad r \leq 1, \end{aligned}$$

and that

$$\sum_{k=1}^{\infty} \frac{\cos k\theta}{k} = -\ln \left| 2 \sin \frac{\theta}{2} \right|, \quad \sum_{k=1}^{\infty} \frac{\sin k\theta}{k} = \begin{cases} \frac{1}{2}(\pi - \theta), & 0 < \theta < \pi, \\ -\frac{1}{2}(\pi + \theta), & -\pi < \theta < 0. \end{cases}$$

Observe that for  $r < 1$  the Fourier coefficients tend to zero faster than exponentially with increasing  $k$ , in contrast to the Fourier coefficients obtained in exercises 8.8 and 8.9, for example.

## 8.6 Addition and multiplication of Fourier series

The Fourier coefficients of the sum and difference of two functions are given by the sum and difference of the constituent coefficients, as would be expected. Thus if

$$f_1(x) = \sum_{k=-\infty}^{\infty} c_k e^{-ikx}, \quad \text{and} \quad f_2(x) = \sum_{k=-\infty}^{\infty} d_k e^{-ikx}, \quad (8.20)$$

then, for any constants  $a$  and  $b$ ,

$$af_1(x) + bf_2(x) = \sum_{k=-\infty}^{\infty} (ac_k + bd_k) e^{-ikx}.$$

The Fourier series of the product of the two functions is, however, more complicated; suppose that

$$f_1(x)f_2(x) = \sum_{k=-\infty}^{\infty} D_k e^{-ikx}, \quad (8.21)$$

then if the Fourier series for  $f_1$  and  $f_2$  are absolutely convergent we also have

$$\begin{aligned} f_1(x)f_2(x) &= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} c_k d_l e^{-i(k+l)x}, \\ &= \sum_{p=-\infty}^{\infty} e^{-ipx} \sum_{k=-\infty}^{\infty} c_k d_{p-k}. \end{aligned} \quad (8.22)$$

On comparing equations 8.21 and 8.22 we see that the  $n$ th Fourier coefficient of the product is

$$D_n = \sum_{k=-\infty}^{\infty} c_k d_{n-k} = \sum_{k=-\infty}^{\infty} c_{n-k} d_k. \quad (8.23)$$

### Exercise 8.13

Show that the  $n$ th Fourier coefficient of  $f_1(x)^2$ , where the Fourier series of  $f_1$  is given in equation 8.20, is

$$\sum_{k=-\infty}^{\infty} c_k c_{n-k}.$$

Use the Fourier series for  $x$  to deduce that

$$\sum_{\substack{k=-\infty \\ k \neq 0, n}}^{\infty} \frac{1}{k(k-n)} = \frac{2}{n^2}.$$

Another consequence of equation 8.22 is the addition formula for Bessel functions, exercise 8.31 (page 295).

## 8.7 The behaviour of Fourier coefficients

In general it is difficult to make a priori estimates of the asymptotic behaviour of Fourier components, although we know from Bessel's inequality that  $\lim_{k \rightarrow \infty} c_k = 0$ , and that  $|c_k|$  must decay to zero faster than  $k^{-1/2}$  if  $f(x)$  is square integrable. For the function treated in exercise 8.8 we have  $c_k = O(k^{-2})$ , and in exercise 8.9  $c_k = O(k^{-1})$ ; exercise 8.12 provides an example for which  $c_k \rightarrow 0$  exponentially. It is clearly important to know how rapidly  $|c_k|$  decreases to zero because this determines the number of terms needed to achieve a given accuracy. Here we present a few elementary observations.

Consider a function having  $N$  continuous derivatives on  $(-\pi, \pi)$ . The integral for the Fourier components, equation 8.13 (page 273), can be integrated by parts  $N$  times; the first two integrations give

$$\begin{aligned} c_k &= \frac{(-1)^k}{2\pi ik} [f(\pi) - f(-\pi)] - \frac{1}{2\pi ik} \int_{-\pi}^{\pi} dx f'(x) e^{ikx} \\ &= \frac{(-1)^k}{2\pi ik} [f(\pi) - f(-\pi)] + \frac{(-1)^k}{2\pi k^2} [f'(\pi) - f'(-\pi)] - \frac{1}{2\pi k^2} \int_{-\pi}^{\pi} dx f''(x) e^{ikx}. \end{aligned} \quad (8.24)$$

Clearly this process can be continued until the  $N$ th differential appears in the integral, but useful information can be gleaned from these expressions.

If  $f(x)$  is even,  $f(\pi) = f(-\pi)$  and it follows that  $c_k = O(k^{-2})$ . The same result holds if  $f$  is not even but  $f(\pi) = f(-\pi)$ . If the function is odd then  $c_k = O(k^{-1})$ , unless  $f(\pi) = 0$ .

If  $f(x)$  is  $2\pi$ -periodic then  $f^{(r)}(\pi) = f^{(r)}(-\pi)$ ,  $r = 0, 1, \dots, N$ , and after further integration by parts we obtain

$$c_k = \frac{1}{2\pi} \left( \frac{i}{k} \right)^N \int_{-\pi}^{\pi} dx f^{(N)}(x) e^{ikx},$$

since all the boundary terms are now zero. But

$$\left| \int_{-\pi}^{\pi} dx f^{(N)}(x) e^{ikx} \right| \leq \left| \int_{-\pi}^{\pi} dx f^{(N)}(x) \right|,$$

so  $c_k = O(k^{-N})$ . If  $f(x)$  is periodic and *all* derivatives exist then  $c_k$  will tend to zero faster than any power of  $1/k$ , for instance as  $e^{-k}$ , as in the example of exercise 8.12.

One important consequence of this last result is that the numerical estimate of the mean of a sufficiently well behaved periodic function over a period obtained using  $N$  equally spaced points converges faster than any power of  $N^{-1}$ ; this is faster than most other numerical procedures. We prove this using the Fourier series of the function: suppose, for simplicity, that the function is  $2\pi$ -periodic so possesses the Fourier series

$$f(x) = \sum_{k=-\infty}^{\infty} C_k e^{-ikx},$$

where the coefficients  $C_k$  are unknown and  $C_0$  is the required mean value of  $f(x)$ . The mean of  $f(x)$  over  $(0, 2\pi)$  can be approximated by the sum over  $N$  equally spaced points,

$$\frac{1}{2\pi} \int_0^{2\pi} dx f(x) \simeq \frac{1}{N} \sum_{j=1}^N f\left(\frac{2\pi j}{N}\right). \quad (8.25)$$

Using the Fourier series of  $f(x)$  the sum can be written in the alternative form

$$\begin{aligned} \sum_{j=1}^N f\left(\frac{2\pi j}{N}\right) &= \sum_{j=1}^N \sum_{k=-\infty}^{\infty} C_k \exp\left(-i\frac{2\pi k j}{N}\right) \\ &= \sum_{k=-\infty}^{\infty} C_k \sum_{j=1}^N \exp\left(-i\frac{2\pi k j}{N}\right). \end{aligned}$$

But we have the relation

$$\sum_{j=1}^N e^{-izj} = R(z) \exp(-i(N+1)z/2), \quad R(z) = \frac{\sin(Nz/2)}{\sin(z/2)},$$

which follows because the left hand side is a geometric series. Now  $z = 2\pi k/N$ , and

$$R\left(\frac{2\pi k}{N}\right) = \frac{\sin \pi k}{\sin(\pi k/N)}.$$

This is zero unless  $k = Np$  for some integer  $p$ , that is  $z = 2\pi p$ , in which case we can find the value of  $R$  by taking the limit, or more easily by noting that the original sum becomes

$$\sum_{j=1}^N \exp\left(-i\frac{2\pi k j}{N}\right) = \sum_{j=1}^N \exp(-i2\pi p j) = N, \quad k = Np.$$

Thus

$$\frac{1}{N} \sum_{j=1}^N f\left(\frac{2\pi j}{N}\right) = C_0 + \sum_{\substack{p=-\infty \\ p \neq 0}}^{\infty} C_{Np}. \quad (8.26)$$

If all derivatives of  $f$  exist then, since  $f(x)$  is periodic,  $|C_{Np}| \rightarrow 0$  faster than any power of  $(Np)^{-1}$ , so that the numerical estimate of the mean on the left hand side converges to  $C_0$  faster than any power of  $N^{-1}$ . This result is of practical value.

The ideas presented in this section can sometimes be put to good use in speeding the convergence of Fourier series. Consider a function  $f(x)$  continuous on  $-\pi < x \leq \pi$  but with  $f(\pi) \neq f(-\pi)$ , so its Fourier coefficients  $c_k = O(k^{-1})$ , as  $k \rightarrow \infty$ . Define a new function  $g(x) = f(x) - \alpha x$  with the constant  $\alpha$  chosen to make  $g(\pi) = g(-\pi)$ , that is

$$\alpha = \frac{1}{2\pi}(f(\pi) - f(-\pi)),$$

so Fourier components of  $g$  behave as  $O(k^{-2})$ .

As an example consider the odd function  $f(x) = \sin \sqrt{2}x$  with the Fourier series

$$\sin \sqrt{2}x = \sum_{k=1}^{\infty} b_k \sin kx, \quad (8.27)$$

where

$$b_k = \frac{2(-1)^{k-1} \sin \pi \sqrt{2}}{\pi(k^2 - 2)} = \frac{2(-1)^{k-1}}{k\pi} \sin \pi \sqrt{2} + O(k^{-2}).$$

In this case  $\alpha = (\sin \pi \sqrt{2})/\pi$  and using the Fourier series of  $x$ , given in the appendix of this chapter, we see that the Fourier coefficients,  $G_k$ , of  $g(x) = f(x) - \alpha x$ , are

$$G_k = b_k - \frac{2(-1)^{k-1}}{k\pi} \sin \pi \sqrt{2} = \frac{4}{\pi} \frac{(-1)^{k-1}}{k(k^2 - 2)} \sin \pi \sqrt{2} = O(k^{-3}).$$

Hence for  $-\pi \leq x \leq \pi$  we may write

$$f(x) = \sin \sqrt{2}x = \frac{x}{\pi} \sin \pi \sqrt{2} + \frac{4}{\pi} \sin \pi \sqrt{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k(k^2 - 2)} \sin kx. \quad (8.28)$$

In the following two figures we compare the function,  $f(x) = \sin \sqrt{2}x$ , with ten terms of the original Fourier series 8.27, on the left, and just two terms of the modified series 8.28, on the right; in the second case with more terms the two functions are practically indistinguishable. The main point to notice is that by removing the discontinuity in the Fourier series at  $x = \pm\pi$  a far more rapidly converging approximation has been obtained that also converges *pointwise* over the whole range.

It is clear that further corrections that produce a more rapidly converging Fourier series may be added. This idea is developed further by Lanczos (1966, section 16).

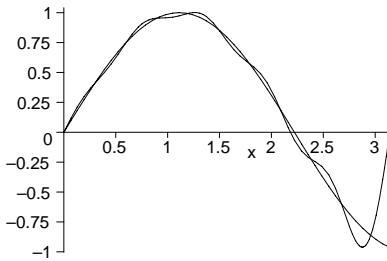


Figure 8.3 Comparison of  $f(x)$  with ten terms of the original Fourier series 8.27.

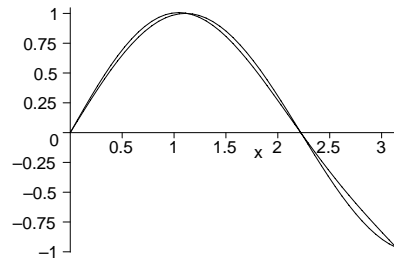


Figure 8.4 Comparison of  $f(x)$  with two terms of the modified Fourier series 8.28.

### Exercise 8.14

Show that the Fourier series of  $f(x) = \sinh x$  on  $(-\pi, \pi)$  is

$$\sinh x = \frac{2 \sinh \pi}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1} k}{1 + k^2} \sin kx.$$

Show further that

$$\sinh x = \frac{x}{\pi} \sinh \pi - 2 \frac{\sinh \pi}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k(1 + k^2)} \sin kx.$$

Use Maple to compare various partial sums of these two series with  $\sinh x$  and hence demonstrate that the latter is a more useful approximation.

## 8.8 Differentiation and integration of Fourier series

The Fourier series of a function  $f(x)$  is uniformly convergent in an interval  $(a, b)$ , where  $-\pi < a \leq x \leq b < \pi$ , if  $f(x)$  is continuous in this interval. Then the term by term differentiation and integration of the series are respectively the differential and integral

of  $f(x)$ . Thus if

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{-ikx} \quad (8.29)$$

is uniformly convergent for  $a \leq x \leq b$ , then

$$\frac{df}{dx} = -i \sum_{k=-\infty}^{\infty} k c_k e^{-ikx}, \quad (8.30)$$

and the integral is

$$\int dx f(x) = A + xc_0 + i \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{c_k}{k} e^{-ikx},$$

where  $A$  is the constant of integration and the sum does not include the  $k = 0$  term. The expressions for the differential and integral of a function given above are not entirely satisfactory: the series for  $\int dx f(x)$  is not a Fourier series, and if  $c_k = O(k^{-1})$  as  $k \rightarrow \infty$  the convergence of the series given for  $f'(x)$  is problematical and the series is certainly not useful.

### Integration

The first difficulty may be cured by forming a definite integral and using the known Fourier series for  $x$ . The definite integral is

$$\int_0^x dx f(x) = xc_0 + i \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{c_k}{k} (e^{-ikx} - 1),$$

but

$$x = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sin kx = i \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{(-1)^{k-1}}{k} e^{-ikx}.$$

Substituting this expression for  $x$  gives

$$\int_0^x dx f(x) = 2 \sum_{k=1}^{\infty} \frac{\Im(c_k)}{k} + i \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \left( \frac{c_k - (-1)^k c_0}{k} \right) e^{-ikx}. \quad (8.31)$$

If  $f(x)$  is real the alternative, real, form for the integral is

$$\int_0^x dx f(x) = \sum_{k=1}^{\infty} \frac{b_k}{k} + \sum_{k=1}^{\infty} \left\{ \frac{a_k - (-1)^k a_0}{k} \sin kx - \frac{b_k}{k} \cos kx \right\}, \quad (8.32)$$

where  $2c_k = a_k + ib_k$ ,  $k \geq 1$  and  $a_0 = 2c_0$ .

**Exercise 8.15**

Use equation 8.32 to integrate the Fourier series for  $x$  to show that

$$x^2 = \frac{\pi^2}{3} - 4 \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cos kx}{k^2}.$$

Explain why the Fourier series for  $x$  on  $(-\pi, \pi)$  depends only on sine functions and that for  $x^2$  only upon cosine functions.

**Differentiation**

For many differentiable functions the leading term in the asymptotic expansion of  $c_k$  is  $O(k^{-1})$ ; the reason for this is given in the previous section, particularly equation 8.24. Then the convergence of the series 8.30 for  $f'(x)$  is questionable even though  $f'(x)$  may have a Fourier series expansion. Normally  $c_k = O(k^{-1})$  because the periodic extension of  $f(x)$  is discontinuous at  $x = \pm\pi$ , then we expect the Fourier coefficients of  $f'(x)$  to also be  $O(k^{-1})$ , not  $O(1)$  as suggested by equation 8.30. We now show how this is achieved.

The analysis of section 8.7 leads us to assume that for large  $k$

$$c_k = (-1)^k \frac{c_{\infty}}{k} + O(k^{-2}), \quad c_{\infty} = \frac{1}{2\pi i} (f(\pi) - f(-\pi)),$$

so we define a constant  $c$  by the limit

$$c = \lim_{k \rightarrow \infty} (-1)^k k c_k$$

and consider the Fourier series

$$g(x) = -i \sum_{k=-\infty}^{\infty} (k c_k - (-1)^k c) e^{-ikx}$$

which is clearly related to the series 8.30. With the assumed behaviour of  $c_k$ , the Fourier components of this series are  $O(k^{-1})$ . Now integrate this function using equation 8.31:

$$\int_0^x du g(u) = f(x) - 2 \sum_{k=1}^{\infty} \frac{1}{k} \Re(k c_k - (-1)^k c).$$

Thus  $g(x) = f'(x)$  and we have found the Fourier series of  $f'(x)$  that converges at the same rate as the Fourier series of  $f(x)$ , that is

$$\frac{df}{dx} = g(x) = -i \sum_{k=-\infty}^{\infty} (k c_k - (-1)^k c) e^{-ikx}. \quad (8.33)$$

If  $f(x)$  is real,  $\Re(c) = 0$  and the real form of the Fourier series is

$$\frac{df}{dx} = -\frac{b}{2} + \sum_{k=1}^{\infty} \{ (k b_k - (-1)^k b) \cos kx - k a_k \sin kx \},$$

where  $b = -2ic = \lim_{k \rightarrow \infty} (-1)^k k b_k$  and  $a_k + i b_k = 2c_k$ .

**Exercise 8.16**

Using the Fourier series

$$x^3 = 2 \sum_{k=1}^{\infty} (-1)^{k-1} \frac{k^2 \pi^2 - 6}{k^3} \sin kx, \quad |x| < \pi,$$

and the above expression for the differential of a Fourier series, show that

$$x^2 = \frac{\pi^2}{3} - 4 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} \cos kx, \quad |x| \leq \pi.$$

Explain why the Fourier coefficients for  $x^3$  and  $x^2$  are respectively  $O(k^{-1})$  and  $O(k^{-2})$  as  $k \rightarrow \infty$ .

**Exercise 8.17**

Consider the function

$$f(x) = \sum_{k=2}^{\infty} (-1)^k \frac{k}{k^2 - 1} \sin kx = \frac{i}{2} \sum_{|k| \geq 2} (-1)^k \frac{k}{k^2 - 1} e^{-ikx}.$$

Show that the Fourier series of  $f'(x)$  is

$$f'(x) = -\frac{1}{2} + \cos x + \sum_{k=2}^{\infty} (-1)^k \frac{\cos kx}{k^2 - 1}.$$

Find also the series for  $f''(x)$  and hence show that  $f(x) = \frac{1}{4} \sin x + \frac{1}{2} x \cos x$ .

**Exercise 8.18**

- (i) If  $f(x) = e^{ax}$ , use the fact that  $f'(x) = af(x)$  together with equation 8.33 to show that the Fourier coefficients of  $f(x)$  on the interval  $(-\pi, \pi)$  are

$$c_k = \frac{i(-1)^k c}{a + ik}$$

for some number  $c$ .

Show directly that the mean of  $f'(x)$  is  $(f(\pi) - f(-\pi))/2\pi$  and use this to determine the value of  $c$ , hence showing that

$$e^{ax} = \frac{\sinh \pi a}{\pi} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{a + ik} e^{-ikx}.$$

- (ii) By observing that  $e^{2ax} = e^{ax} e^{ax}$ , or otherwise, use the Fourier series for  $e^{ax}$  to show that

$$1 + 2a^2 \sum_{k=1}^{\infty} \frac{1}{a^2 + k^2} = \frac{\pi a}{\tanh \pi a},$$

and by considering the small  $a$  expansion deduce that, for some numbers  $f_n$ ,

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = f_n \pi^{2n}, \quad n = 1, 2, \dots$$

### 8.9 Fourier series on arbitrary ranges

The Fourier series of a function  $f(x)$  over a range  $a \leq x \leq b$ , different from  $(-\pi, \pi)$ , is obtained using the preceding results and by defining a new variable

$$w = \frac{2\pi}{b-a} \left( x - \frac{b+a}{2} \right), \quad x = \frac{b+a}{2} + \frac{b-a}{2\pi} w, \quad (8.34)$$

which maps  $-\pi \leq w \leq \pi$  onto  $a \leq x \leq b$ . Then it follows that the function  $f(x(w))$  of  $w$  can be expressed as a Fourier series on  $-\pi \leq w \leq \pi$ ,

$$f(x(w)) = \sum_{k=-\infty}^{\infty} c_k e^{-ikw} \quad \text{and so} \quad f(x) = \sum_{k=-\infty}^{\infty} d_k \exp \left( -i \frac{2\pi k}{b-a} x \right),$$

where

$$d_k = \frac{1}{b-a} \int_a^b dx f(x) \exp \left( \frac{2\pi i k}{b-a} x \right), \quad (8.35)$$

this last relation following from the definition of  $c_k$  and the transformation 8.34.

#### Exercise 8.19

Show that the Fourier series of  $f(x) = \cosh x$  on the interval  $(-L, L)$  is

$$\cosh x = \frac{\sinh L}{L} \left( 1 + 2 \sum_{k=1}^{\infty} \frac{(-1)^k L^2}{L^2 + \pi^2 k^2} \cos \left( \frac{\pi k x}{L} \right) \right), \quad -L \leq x \leq L.$$

From the form of this Fourier series we observe that the Fourier components are small only if  $k \gg L$ ; thus if  $L$  is large many terms of the series are needed to provide an accurate approximation, as may be seen by comparing graphs of partial sums of the series with  $\cosh x$  on  $-L < x < L$  for various values of  $L$ .

### 8.10 Sine and cosine series

Fourier series of even functions,  $f(x) = f(-x)$ , comprise only cosine terms, and Fourier series of odd functions,  $f(x) = -f(-x)$ , comprise only sine terms. This fact can be used to produce cosine or sine series of *any* function over a given range which we shall take to be  $(0, \pi)$ .

For any function  $f(x)$  an odd extension,  $f_o(x)$ , may be produced as follows:

$$f_o(x) = \begin{cases} -f(-x), & x < 0, \\ f(x), & x \geq 0. \end{cases}$$

Figure 8.6 shows the odd extension of the function  $f(x)$ , depicted in figure 8.5.

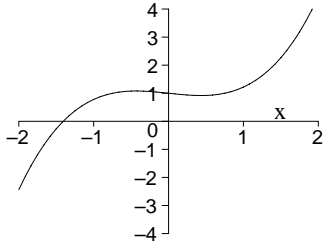


Figure 8.5 Original function.

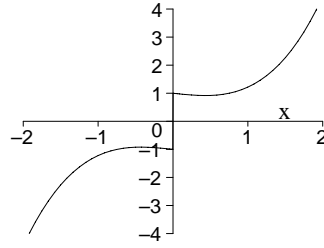


Figure 8.6 Odd extension.

We would normally use this extension when  $f(0) = 0$  so that  $f_o(x)$  is continuous at  $x = 0$ . Then the Fourier series of  $f_o(x)$  on  $(-\pi, \pi)$  contains only sine functions and we have, from equation 8.15,

$$f(x) = \sum_{k=1}^{\infty} b_k \sin kx, \quad 0 < x \leq \pi, \quad \text{with} \quad b_k = \frac{2}{\pi} \int_0^{\pi} dx f(x) \sin kx, \quad k = 1, 2, \dots,$$

the latter relation following from equation 8.13. This series is often called the *half-range Fourier sine series*.

Similarly we can extend  $f(x)$  to produce an even function,

$$f_e(x) = \begin{cases} f(-x), & x < 0, \\ f(x), & x \geq 0, \end{cases}$$

an example of which is shown in figure 8.8.

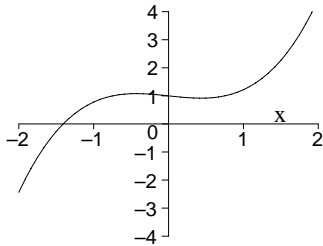


Figure 8.7 Original function.

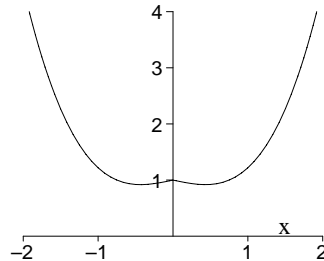


Figure 8.8 Even extension.

The even extension produces a Fourier series containing only cosine functions, to give the *half-range Fourier cosine series*

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos kx, \quad 0 < x \leq \pi,$$

where

$$a_k = \frac{2}{\pi} \int_0^{\pi} dx f(x) \cos kx, \quad k = 0, 1, \dots$$

**Exercise 8.20**

Show that

$$x = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sin kx, \quad 0 \leq x < \pi,$$

and

$$x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\cos(2k-1)x}{(2k-1)^2}, \quad 0 \leq x \leq \pi.$$

Explain why the even extension converges more rapidly.

Use Maple to compare, graphically, the partial sums of these two approximations with the exact function.

**Exercise 8.21**

Show that

$$\sin x = \frac{4}{\pi} \left\{ \frac{1}{2} - \sum_{k=1}^{\infty} \frac{\cos 2kx}{4k^2 - 1} \right\}, \quad 0 \leq x \leq \pi,$$

and deduce that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{4k^2 - 1} = \frac{\pi}{4} - \frac{1}{2}.$$

**8.11 The Gibbs phenomenon**

The Gibbs phenomenon occurs in the Fourier series of any discontinuous function, such as that considered in exercise 8.9, but before discussing the mathematics behind this strange behaviour we give the following description of its discovery, taken from Lanczos (1966, section 10).

The American physicist Michelson† invented many physical instruments of very high precision. In 1898 he constructed a harmonic analyser that could compute the first 80 Fourier components of a function described numerically; this machine could also construct the graph of a function from the Fourier components, thus providing a check on the operation of the machine because, having obtained the Fourier components from a given function, it could be reconstructed and compared with the original. Michelson found that in most cases the input and synthesised functions agreed well.

† A. A. Michelson (1852–1931) was born in Prussia. He moved to the USA when two years old and is best known for his accurate measurements of the speed of light, which was his life long passion; in 1881 he determined this to be 186 329 miles/sec and in his last experiment, finished after his death, obtained 186 280 miles/sec, the current value for the speed of light in a vacuum is 186 282.397 miles/sec ( $2.997\,924\,58 \times 10^8$  m/sec). In 1887, together with Morley, he published results showing that there was no measurable difference between the speed of light when the Earth was moving towards or away from a source: this result was crucial in falsifying the theories of the aether and in Einstein's formulation of the special theory of relativity.